

# Fermionic one- and two-dimensional Toda lattice hierarchies and their bi-Hamiltonian structures

V.V. Gribanov<sup>1</sup>, V.G. Kadyshevsky,<sup>2</sup> and A.S. Sorin<sup>3</sup>

*Bogoliubov Laboratory of Theoretical Physics,  
Joint Institute for Nuclear Research,  
141980 Dubna, Moscow Region, Russia*

## Abstract

By exhibiting the corresponding Lax pair representations we propose a wide class of integrable two-dimensional (2D) fermionic Toda lattice (TL) hierarchies which includes the 2D  $N = (2|2)$  and  $N = (0|2)$  supersymmetric TL hierarchies as particular cases. Performing their reduction to the one-dimensional case by imposing suitable constraints we derive the corresponding 1D fermionic TL hierarchies. We develop the generalized graded R-matrix formalism using the generalized graded bracket on the space of graded operators with an involution generalizing the graded commutator in superalgebras, which allows one to describe these hierarchies in the framework of the Hamiltonian formalism and construct their first two Hamiltonian structures. The first Hamiltonian structure is obtained for both bosonic and fermionic Lax operators while the second Hamiltonian structure is established for bosonic Lax operators only. We propose the graded modified Yang-Baxter equation in the operator form and demonstrate that for the class of graded antisymmetric  $R$ -matrices it is equivalent to the tensor form of the graded classical Yang-Baxter equation.

*PACS:* 02.20.Sv; 02.30.Jr; 11.30.Pb

*Keywords:* Completely integrable systems; Toda field theory; Supersymmetry; Discrete symmetries; R-matrix

*E-mail:*

1) [gribanov@thsun1.jinr.ru](mailto:gribanov@thsun1.jinr.ru)

2) [kadyshev@jinr.ru](mailto:kadyshev@jinr.ru)

3) [sorin@thsun1.jinr.ru](mailto:sorin@thsun1.jinr.ru)

# 1 Introduction

The Toda lattice (TL) is one of the most important families of models in the theory of integrable systems. Its various generalizations and supersymmetric extensions, having deep implications in modern mathematical physics, have been the subject of intense investigations during the last decades.

The 2D TL hierarchy was first studied in [1, 2], and at present two different nontrivial supersymmetric extensions of 2D TL are known. They are the  $N = (2|2)$  [3]-[12] and  $N = (0|2)$  [12, 9] supersymmetric TL hierarchies that possess a different number of supersymmetries and contain the  $N = (2|2)$  and  $N = (0|2)$  TL equations as subsystems. Quite recently, the 2D generalized fermionic TL equations have been introduced [11] and their two reductions related to the  $N = (2|2)$  and  $N = (0|2)$  supersymmetric TL equations were considered. In the present paper, we describe a wide class of integrable two-dimensional fermionic Toda lattice hierarchies – 2D fermionic  $(K^+, K^-)$ -TL hierarchies, which includes the 2D  $N = (2|2)$  and  $N = (0|2)$  supersymmetric TL hierarchies as particular cases and contains the 2D generalized fermionic TL equations as a subsystem.

The Hamiltonian description of the 2D TL hierarchy has been constructed only quite recently in the framework of the R-matrix approach in [13], where the new R-matrix associated with splitting of algebra given by a pair of difference operators was introduced. In the present paper, we adapt this R-matrix to the case of  $Z_2$ -graded operators and derive the bi-Hamiltonian structure of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy.

Remarkably, in solving this problem the generalized graded bracket (5) on the space of graded operators with an involution finds its new application. This bracket was introduced in [14], where it was observed that the  $N = (1|1)$  supersymmetric 2D TL hierarchy had a natural Lax-pair representation in terms of this bracket which allowed one to derive the dispersionless  $N = (1|1)$  2D TL hierarchy and its Lax representation. In the present paper, the generalized graded bracket is used to describe the 2D fermionic  $(K^+, K^-)$ -TL hierarchy and define its two Hamiltonian structures. Moreover, we demonstrate that the classical graded Yang-Baxter equation [15] has an equivalent operator representation in terms of the generalized graded bracket. All these facts attest that this bracket has a fundamental meaning and allows a broad spectrum of applications in modern mathematical physics.

This paper is the extended version of [16]. The structure of the paper is as follows.

In Sec. 2, we define the space of the  $Z_2$ -graded difference operators with the involution and recall the generalized graded bracket [14] and its properties.

In Sec. 3, we give a theoretical background of the  $R$ -matrix method generalized to the case of the  $Z_2$ -graded difference operators. We define the  $R$ -matrix on the associative algebra  $\mathfrak{g}$  of the  $Z_2$ -graded difference operators, derive the graded modified Yang-Baxter equation and using the generalized graded bracket obtain two Poisson brackets for the functionals on  $\mathfrak{g}^\dagger = \mathfrak{g}$ . The proper properties of the Poisson brackets thus obtained are provided by the properties of the generalized graded bracket. Thus, for the  $Z_2$ -graded difference operators of odd (even) parity this bracket defines odd (even) first Poisson bracket. The second Poisson bracket is found only for even difference operators which in this case are compatible with the first Poisson bracket. Using these Poisson brackets one can define the Hamiltonian equations that can equivalently

be rewritten in terms of the Lax-pair representation. The basic results of Sec. 3 are formulated as Theorem.

In Sec. 4, using the generalized graded bracket we propose a new 2D fermionic  $(K^+, K^-)$ -TL hierarchy in terms of the Lax-pair representation and construct the algebra of its flows. Then we present the explicit expression for its flows and show that all known up to now 2D TL equations can be derived from this hierarchy as subsystems.

In Sec. 5, we consider the reduction of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy to the 1D space and reproduce the 1D generalized fermionic TL equations [11] as the first flow of the reduced hierarchy with additional constraint imposed.

In Secs. 6 and 7, we apply the results of Sec. 3 to derive the Hamiltonian structures of the 1D and 2D fermionic  $(K^+, K^-)$ -TL hierarchies. Following [13] we use the  $R$ -matrix which acts nontrivially on the space of the direct sum of two difference operators and derive two different Hamiltonian structures of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy. The first Hamiltonian structure is obtained for both even and odd values of  $(K^+, K^-)$  while the second one is found for even values of  $(K^+, K^-)$  only. We perform their Dirac reduction and demonstrate that in general the Dirac brackets for the second Hamiltonian structure are nonlocal but for the case of the fermionic  $(2, 2)$ -TL hierarchy they become local. As an example, we give the explicit form of the first and second Hamiltonian structures for the fermionic 1D  $(2, 2)$ -TL hierarchy.

In Sec. 8, we briefly summarize the main results obtained in this paper and point out open problems. In Appendices, we clarify some technical aspects.

## 2 Space of difference operators

In this section, we define the space of difference operators which will play an important role in our consideration. These operators can be represented in the following general form:

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} e^{(k-m)\partial}, \quad m, j \in \mathbb{Z}, \quad (1)$$

parameterized by the functions  $f_{2k,j}^{(m)}$  ( $f_{2k+1,j}^{(m)}$ ) which are the  $Z_2$ -graded bosonic (fermionic) lattice fields with the lattice index  $j$  ( $j \in \mathbb{Z}$ ) and the Grassmann parity defined by index  $k$

$$d_{f_{k,j}^{(m)}} = |k| \bmod 2.$$

$e^{k\partial}$  is the shift operator whose action on the lattice fields results into a discrete shift of a lattice index

$$e^{l\partial} f_{k,j}^{(m)} = f_{k,j+l}^{(m)} e^{l\partial}. \quad (2)$$

The shift operator has  $Z_2$ -parity defined as

$$d'_{e^{l\partial}} = |l| \bmod 2.$$

The operators  $\mathbb{O}_m$  (1) admit the diagonal  $Z_2$ -parity

$$d_{\mathbb{O}_m} = d_{f_{k,j}^{(m)}} + d'_{e^{(k-m)\partial}} = |m| \bmod 2 \quad (3)$$

and the involution

$$\mathbb{O}_m^* = \sum_{k=-\infty}^{\infty} (-1)^k f_{k,j}^{(m)} e^{(k-m)\partial}.$$

In what follows we also need the projections of the operators  $\mathbb{O}_m$  defined as

$$(\mathbb{O}_m)_{\leq p} = \sum_{k \leq p+m} f_{k,j} e^{(k-m)\partial}, \quad (\mathbb{O}_m)_{\geq p} = \sum_{k \geq p+m} f_{k,j} e^{(k-m)\partial}$$

and we will use the usual notation for the projections  $(\mathbb{O}_m)_+ := (\mathbb{O}_m)_{\geq 0}$  and  $(\mathbb{O}_m)_- := (\mathbb{O}_m)_{< 0}$ . Note that  $e^{l\partial}$  is a conventional form for the shift operators defined in terms of infinite-dimensional matrices  $(e^{l\partial})_{i,j} \equiv \delta_{i,j-l}$ , and there is an isomorphism between operators (1) and infinite-dimensional matrices (see e.g. [17])

$$\mathbb{O}_m = \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} e^{(k-m)\partial} \rightarrow (\mathbb{O}_m)_{j,i} \equiv \sum_{k=-\infty}^{\infty} f_{k,j}^{(m)} \delta_{j,i-k+m}.$$

In the operator space (1) one can extract two subspaces which are of great importance in our further consideration

$$\mathbb{O}_{K^\pm}^\pm = \sum_{k=0}^{\infty} f_{k,j}^{\pm} e^{\pm(K^\pm-k)\partial}, \quad K^\pm \in \mathbb{N}. \quad (4)$$

The operators of the subspaces  $\mathbb{O}_{K^\pm}^\pm$  form associative algebras with the multiplication (2). Using this fact we define on these subspaces the generalized graded algebra with the bracket [14]

$$[\mathbb{O}, \tilde{\mathbb{O}}] := \mathbb{O} \tilde{\mathbb{O}} - (-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} \tilde{\mathbb{O}}^{*(d_{\mathbb{O}})} \mathbb{O}^{*(d_{\tilde{\mathbb{O}}})}, \quad (5)$$

where the operators  $\mathbb{O}$  and  $\tilde{\mathbb{O}}$  belong to the subspaces  $\mathbb{O}_{K^+}^+$  ( $\mathbb{O}_{K^-}^-$ ), and  $\mathbb{O}^{*(m)}$  denotes the  $m$ -fold action of the involution  $*$  on the operator  $\mathbb{O}$ , ( $\mathbb{O}^{*(2)} = \mathbb{O}$ ). Bracket (5) generalizes the (anti)commutator in superalgebras and satisfies the following properties [14]:

symmetry

$$[\mathbb{O}, \tilde{\mathbb{O}}] = -(-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} [\tilde{\mathbb{O}}^{*(d_{\mathbb{O}})}, \mathbb{O}^{*(d_{\tilde{\mathbb{O}}})}], \quad (6)$$

derivation

$$[\mathbb{O}, \tilde{\mathbb{O}} \hat{\mathbb{O}}] = [\mathbb{O}, \tilde{\mathbb{O}}] \hat{\mathbb{O}} + (-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} \tilde{\mathbb{O}}^{*(d_{\mathbb{O}})} [\mathbb{O}^{*(d_{\tilde{\mathbb{O}}})}, \hat{\mathbb{O}}], \quad (7)$$

and Jacobi identity

$$\begin{aligned} (-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} [[\mathbb{O}, \tilde{\mathbb{O}}^{*(d_{\mathbb{O}}})], \hat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\tilde{\mathbb{O}}})}] + (-1)^{d_{\tilde{\mathbb{O}}} d_{\mathbb{O}}} [[\tilde{\mathbb{O}}, \hat{\mathbb{O}}^{*(d_{\tilde{\mathbb{O}}})}], \mathbb{O}^{*(d_{\tilde{\mathbb{O}}+d_{\mathbb{O}}})}] \\ + (-1)^{d_{\tilde{\mathbb{O}}} d_{\hat{\mathbb{O}}}} [[\hat{\mathbb{O}}, \mathbb{O}^{*(d_{\hat{\mathbb{O}}})}], \tilde{\mathbb{O}}^{*(d_{\hat{\mathbb{O}}+d_{\mathbb{O}}})}] = 0. \end{aligned} \quad (8)$$

For the operators  $\mathbb{O}_m$  (1) we define the supertrace

$$str \mathbb{O} = \sum_{j=-\infty}^{\infty} (-1)^j f_{m,j}^{(m)}. \quad (9)$$

In what follows we assume suitable boundary conditions for the functions  $f_{k,j}^{(m)}$  in order the main property of supertraces

$$str[\mathbb{O}, \tilde{\mathbb{O}}] = 0 \quad (10)$$

be satisfied for the case of the generalized graded bracket (5).

### 3 R-matrix formalism

In this section, we develop a theoretical background of the R-matrix method adapted to the case of the operator space (1).

Let  $\mathfrak{g}$  be an associative algebra of the operators from the space (1) with the invariant non-degenerate inner product

$$\langle \mathbb{O}, \tilde{\mathbb{O}} \rangle = str(\mathbb{O} \tilde{\mathbb{O}})$$

using which one can identify the algebra  $\mathfrak{g}$  with its dual  $\mathfrak{g}^\dagger$ . We set the following Poisson bracket:

$$\{f, g\}(\mathbb{O}) = - \langle \mathbb{O}, [\nabla g, (\nabla f)^*(\nabla d_g)] \rangle, \quad (11)$$

where  $f, g$  are functionals on  $\mathfrak{g}$ , and  $\nabla f$  and  $\nabla g$  are their gradients at the point  $\mathbb{O}$  which are related with  $f, g$  through the inner product

$$\left. \frac{\partial f(\mathbb{O} + \epsilon \delta \mathbb{O})}{\partial \epsilon} \right|_{\epsilon=0} = \langle \delta \mathbb{O}, \nabla f(\mathbb{O}) \rangle.$$

Note that the proper properties of the Poisson bracket (11) follow from the properties (6–8) of the generalized bracket (5) and are strictly determined by the  $Z_2$ -parity of the operator  $\mathbb{O}$ . Thus, one has symmetry

$$\{f, g\} = -(-1)^{(d_f+d_{\mathbb{O}})(d_g+d_{\mathbb{O}})} \{g, f\}, \quad (12)$$

derivation

$$\{f, gh\} = \{f, g\}h + (-1)^{d_g(d_f+d_{\mathbb{O}})} g\{f, h\}, \quad (13)$$

and Jacobi identity

$$\begin{aligned} & (-1)^{(d_f+d_{\mathbb{O}})(d_h+d_{\mathbb{O}})} \{\{f, g\}, h\} + (-1)^{(d_g+d_{\mathbb{O}})(d_f+d_{\mathbb{O}})} \{\{g, h\}, f\} \\ & + (-1)^{(d_h+d_{\mathbb{O}})(d_g+d_{\mathbb{O}})} \{\{h, f\}, g\} = 0. \end{aligned} \quad (14)$$

Therefore, for the even operator  $\mathbb{O}$  one has a usual (even)  $Z_2$ -graded Poisson bracket, while for the operators with odd diagonal parity  $d_{\mathbb{O}}$  (3) eq. (11) defines the odd  $Z_2$ -graded Poisson bracket (antibracket).

Having defined the Poisson bracket we proceed with the search for the hierarchy of flows generated by this bracket using Hamiltonians. Therefore, we need to determine an infinite set of functionals which should be in involution to play the role of Hamiltonians. For Poisson bracket (11) one can find an infinite set of Hamiltonians in a rather standard way

$$H_k = \frac{1}{k} \text{str} \mathbb{O}_*^k = \frac{1}{k} \sum_{i=-\infty}^{\infty} (-1)^i f_{km,i}^{(km)}, \quad (15)$$

where  $\mathbb{O}_*^k$  is defined as

$$(\mathbb{O})_*^{2k} := (\mathbb{O}^{*(d_{\mathbb{O}})} \mathbb{O})^k, \quad (\mathbb{O})_*^{2k+1} := \mathbb{O} (\mathbb{O})_*^{2k}. \quad (16)$$

For the odd operators  $\mathbb{O}$  eq. (15) defines only fermionic nonzero functionals  $H_{2k+1}$ , since in this case even powers of the operators  $\mathbb{O}$  have the following representation:

$$d_{\mathbb{O}} = 1 : \quad (\mathbb{O})_*^{2k} = (1/2[(\mathbb{O})^*, \mathbb{O}])^k \equiv 1/2[(\mathbb{O})_*^{2k-1})^*, \mathbb{O}] \quad (17)$$

and all the bosonic Hamiltonians are trivial ( $H_{2k} = 0$ ) like the supertrace of the generalized graded bracket.

The functionals (15) are obviously in involution but produce a trivial dynamics. Actually, the functionals  $H_k$  (15) are the Casimir operators of the Poisson bracket (11), so the Poisson bracket of  $H_k$  with any other functional is equal to zero as an output (due to the relation  $\nabla H_{k+1} = \mathbb{O}_*^k$ ). Nevertheless, it is possible to modify the Poisson bracket (11) in such a way that the new Poisson bracket would produce nontrivial equations of motion using the same Hamiltonians (15) and these Hamiltonians are in involution with respect to the modified Poisson bracket as well. Let us introduce the modified generalized graded bracket on the space (1)

$$[\mathbb{O}, \tilde{\mathbb{O}}]_R := [R(\mathbb{O}), \tilde{\mathbb{O}}] + [\mathbb{O}, R(\tilde{\mathbb{O}})], \quad (18)$$

where the  $R$ -matrix is a linear map  $R: \mathfrak{g} \rightarrow \mathfrak{g}$  such that the bracket (18) satisfies the properties (6–8). One can verify that the Jacobi identities (8) for the bracket (18) can equivalently be rewritten in terms of the generalized graded bracket (5)

$$\begin{aligned} & (-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} [[\mathbb{O}, \tilde{\mathbb{O}}^{*(d_{\mathbb{O}})}]_R, \hat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\tilde{\mathbb{O}}})}]_R + \text{cycle perm.} = \\ & (-1)^{d_{\mathbb{O}} d_{\tilde{\mathbb{O}}}} [R([\mathbb{O}, \tilde{\mathbb{O}}^{*(d_{\mathbb{O}})}]_R) - [R(\mathbb{O}), R(\tilde{\mathbb{O}}^{*(d_{\mathbb{O}})})], \hat{\mathbb{O}}^{*(d_{\mathbb{O}}+d_{\tilde{\mathbb{O}}})}] + \text{cycle perm.} = 0. \end{aligned}$$

Thus, one can conclude that a sufficient condition for  $R$  to be the  $R$ -matrix is the validity of the following equation:

$$R([\mathbb{O}, \tilde{\mathbb{O}}]_R) - [R(\mathbb{O}), R(\tilde{\mathbb{O}})] = \alpha [\mathbb{O}, \tilde{\mathbb{O}}], \quad (19)$$

where  $\alpha$  is an arbitrary constant. One can show (see Appendix A) that for the case of graded antisymmetric operators  $R$  eq. (19) at  $\alpha = 1$  represents the operator form of the graded

classical Yang-Baxter equation [15]. Following the terminology of [18] we call eq. (19) the graded modified Yang-Baxter equation. Equation (19) is the generalization of the graded modified classical Yang-Baxter equation discussed in paper [19] for the case of space of graded operators (1).

With the new bracket (18) one can define the corresponding new Poisson bracket on dual  $\mathfrak{g}^\dagger$ :

$$\begin{aligned}\{f, g\}_1(\mathbb{O}) &= -1/2 \langle \mathbb{O}, [\nabla g, (\nabla f)^{*(d_{\nabla g})}]_R \rangle \\ &\equiv \frac{1}{2} \langle (-1)^{d_{\nabla g} d_{\mathbb{O}}} R(\nabla g)^{*(d_{\mathbb{O}})} [\mathbb{O}^{*(d_{\nabla g})}, (\nabla f)^{*(d_{\nabla g})}] \rangle \\ &\quad - [\mathbb{O}, \nabla g] R((\nabla f)^{*(d_{\nabla g})}) \rangle .\end{aligned}\tag{20}$$

With respect to the dependence of the r.h.s of (20) on the point  $\mathbb{O}$  this is a linear bracket. Without going into details we introduce also bi-linear bracket for bosonic graded operators  $\mathbb{O}_B$  ( $d_{\mathbb{O}_B} = 0$ ) as follows:

$$\begin{aligned}\{f, g\}_2(\mathbb{O}_B) &= -1/4 \langle [\mathbb{O}_B, \nabla g] R((\nabla f)^{*(d_{\nabla g})} \mathbb{O}_B^{*(d_{\nabla f} + d_{\nabla g})} \\ &\quad + \mathbb{O}_B^{*(d_{\nabla g})} (\nabla f)^{*(d_{\nabla g})}) - R(\nabla g \mathbb{O}_B^{*(d_{\nabla g})}) \\ &\quad + \mathbb{O}_B \nabla g [\mathbb{O}_B^{*(d_{\nabla g})}, (\nabla f)^{*(d_{\nabla g})}] \rangle .\end{aligned}\tag{21}$$

We did not succeed in constructing the bi-linear bracket for the case of fermionic operators  $\mathbb{O}_F$  ( $d_{\mathbb{O}_F} = 1$ ). The bracket (20) is obviously the Poisson bracket if  $R$  is an  $R$ -matrix on  $\mathfrak{g}$ . The bi-linear bracket (21) becomes Poisson bracket under more rigorous constraints which can be found in the following

- Theorem.** a) Linear bracket (20) is the Poisson bracket if  $R$  obeys the graded modified Yang-Baxter equation (19);  
b) the bi-linear bracket (21) is the Poisson bracket if  $R$  and its graded antisymmetric part  $1/2(R - R^\dagger)$  obey the graded modified Yang-Baxter equation (19) with the same  $\alpha$ ;  
c) if  $\mathbb{O} = \mathbb{O}_B$ , then these two Poisson brackets are compatible;  
d) the Casimir operators  $H_k$  (15) of the bracket (11) are in involution with respect to both linear (20) and bi-linear (21) Poisson brackets;  
e) the Hamiltonians  $H_k \neq 0$  (15) generate evolution equations

$$\begin{aligned}\partial_k \mathbb{O} &= \{H_{k+1}, \mathbb{O}\}_1 = 1/2 [R((\nabla H_{k+1})^{*(d_{\mathbb{O}})}), \mathbb{O}], \\ \partial_k \mathbb{O}_B &= \{H_k, \mathbb{O}_B\}_2 = 1/4 [R(\nabla H_k \mathbb{O}_B + \mathbb{O}_B \nabla H_k), \mathbb{O}_B]\end{aligned}\tag{22}$$

via the brackets (20) and (21), respectively, which connect the Lax-pair and Hamiltonian representations.

**Proof.** a). This is a summary of the above discussion on the linear bracket.

b). Using the property of cyclic permutations inside the supertrace one can easily verify that the symmetry property  $\{f, g\}_2 = -(-1)^{d_f d_g} \{g, f\}_2$  holds. Verification of the Jacobi identities for the bi-linear bracket amounts to straightforward and tedious calculations which are presented in Appendix B.

c). These two Poisson brackets are obviously compatible. Indeed, a deformation of the point

$\mathbb{O}_B \longrightarrow \mathbb{O}_B + b$  on the dual  $\mathfrak{g}^\dagger$ , where  $b$  is an arbitrary constant operator, transforms (21) into the sum of two Poisson brackets

$$\{f, g\}_2(\mathbb{O}_B + b) = \{f, g\}_2(\mathbb{O}_B) + b\{f, g\}_1(\mathbb{O}_B).$$

d). Substituting the expressions for Hamiltonians (15) into eqs. (20) and (21) and taking into account that  $\nabla H_{k+1} = \mathbb{O}_*^k$  it is easily to check that the Casimirs of the bracket (11) are in involution with respect to both the Poisson structures (20) and (21).

e). Using cyclic permutations inside the supertrace operation let us rewrite both the Poisson brackets (20) and (21) in the following general form:

$$\{f, g\}_i(\mathbb{O}) = \langle P_i(\mathbb{O}) \nabla g, (\nabla f)^{*(d_{\nabla g})} \rangle, \quad i = 1, 2,$$

where  $P_i(\mathbb{O})$  is the Poisson tensor corresponding to the bracket  $\{\dots, \dots\}_i$

$$\begin{aligned} P_1(\mathbb{O}) \nabla g &= -1/2([\mathbb{O}, R(\nabla g)] + R^\dagger([\mathbb{O}, \nabla g])), \\ P_2(\mathbb{O}_B) \nabla g &= 1/4([R(\nabla g \mathbb{O}_B + \mathbb{O}_B \nabla g), \mathbb{O}_B^{*(d_{\nabla g})}] \\ &\quad - \mathbb{O}_B R^\dagger([\mathbb{O}_B, \nabla g]) - R^\dagger([\mathbb{O}_B, \nabla g]) \mathbb{O}_B^{*(d_{\nabla g})}) \end{aligned}$$

and the adjoint operator  $R^\dagger$  acts on the dual  $\mathfrak{g}^\dagger$

$$\langle \mathbb{O}, R(\tilde{\mathbb{O}}) \rangle = \langle R^\dagger(\mathbb{O}), \tilde{\mathbb{O}} \rangle.$$

The Hamiltonian vector field associated with Hamiltonian  $H_k$  is given by  $\partial_k \mathbb{O} = P_i(\mathbb{O}) \nabla H_k$ . Taking into account that  $[\mathbb{O}, \nabla H_k] = 0$  we arrive at the Lax-pair representations (22).  $\blacksquare$

Note that a similar Theorem when the shift operators and functions parameterizing the difference operators  $\mathbb{O}$  (1) have even  $Z_2$ -parity was discussed in [18, 20, 21, 13].

For the graded modified Yang-Baxter equation (19) there is a particular class of solutions which are useful in application. Suppose that the algebra  $\mathfrak{g}$  can be represented as a vector space direct sum of two subalgebras

$$\mathfrak{g} = \mathfrak{g}_+ \dot{+} \mathfrak{g}_- : \quad [\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_-.$$

Let  $P_\pm$  be the projection operators on these subalgebras,  $P_\pm \mathfrak{g} = \mathfrak{g}_\pm$ , then one can easily verify that  $R = P_+ - P_-$  satisfies the graded modified Yang-Baxter equation (19) at  $\alpha = 1$  and, therefore, represents the  $R$ -matrix on  $\mathfrak{g}$ . Indeed, in this case the modified generalized graded bracket (18)

$$[\mathbb{O}, \tilde{\mathbb{O}}]_R = 2[(\mathbb{O})_+, (\tilde{\mathbb{O}})_+] - 2[(\mathbb{O})_-, (\tilde{\mathbb{O}})_-] \quad (23)$$

obviously satisfies the Jacobi identities (8), since it determines the usual direct sum of two subalgebras

$$[\mathfrak{g}_\pm, \mathfrak{g}_\pm]_R \subset \mathfrak{g}_\pm, \quad [\mathfrak{g}_+, \mathfrak{g}_-]_R = 0.$$



## 4 2D fermionic $(K^+, K^-)$ -Toda lattice hierarchy

In this section, we introduce the two-dimensional fermionic  $(K^+, K^-)$ -Toda lattice hierarchy in terms of the Lax-pair representation.

Let us consider two difference operators  $L_{K^\pm}^\pm$

$$L_{K^+}^+ = \sum_{k=0}^{\infty} u_{k,i} e^{(K^+ - k)\partial}, \quad L_{K^-}^- = \sum_{k=0}^{\infty} v_{k,i} e^{(k - K^-)\partial}, \quad (24)$$

which obviously belong to the spaces (4). The lattice fields and the shift operator entering into these operators have the following length dimensions:  $[u_{k,i}] = -1/2k$ ,  $[v_{k,i}] = 1/2(k - K^+ - K^-)$  and  $[e^{k\partial}] = -1/2k$ , respectively, so operators (24) are of equal length dimension,  $[L_{K^+}^+] = [L_{K^-}^-] = -1/2K^+$ . The dynamics of the fields  $u_{k,i}, v_{k,i}$  are governed by the Lax equations expressed in terms of the generalized graded bracket (5) [14]

$$D_s^\pm L_{K^\alpha}^\alpha = \mp \alpha (-1)^{sK^\alpha K^\pm} [(((L_{K^\pm}^\pm)_*)^s)_{-\alpha}]^{*(K^\alpha)}, L_{K^\alpha}^\alpha\}, \quad \alpha = +, -, \quad s \in \mathbb{N}, \quad (25)$$

where  $D_s^\pm$  are evolution derivatives with the  $Z_2$ -parity defined as

$$d_{D_s^\pm} = sK^\pm \pmod{2}$$

and the length dimension  $[D_s^+] = [D_s^-] = -sK^+/2$ . The Lax equations (25) generate non-Abelian (super)algebra of flows of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy

$$[D_s^\pm, D_p^\pm] = (1 - (-1)^{spK^\pm}) D_{s+p}^\pm, \quad [D_s^+, D_p^-] = 0.$$

The composite operators  $(L_{K^\pm}^\pm)_*^s$  entering into the Lax equations (25) are defined by eq. (16) and also belong to the spaces (4)

$$(L_{K^+}^+)_*^r := \sum_{k=0}^{\infty} u_{k,i}^{(r)} e^{(rK^+ - k)\partial}, \quad (L_{K^-}^-)_*^r := \sum_{k=0}^{\infty} v_{k,i}^{(r)} e^{(k - rK^-)\partial}.$$

Here  $u_{k,i}^{(r)}$  and  $v_{k,i}^{(r)}$  are functionals of the original fields and there are the following recursion relations for them

$$\begin{aligned} u_{p,i}^{(r+1)} &= \sum_{k=0}^p (-1)^{kK^+} u_{k,i}^{(r)} u_{p-k,i-k+rK^+}, & u_{p,i}^{(1)} &= u_{p,i}, \\ v_{p,i}^{(r+1)} &= \sum_{k=0}^p (-1)^{kK^-} v_{k,i}^{(r)} v_{p-k,i+k-rK^-}, & v_{p,i}^{(1)} &= v_{p,i}. \end{aligned}$$

Now using the Lax representation (25) and relations (7) and (16) one can derive the equations of motion for the composite Lax operators

$$D_s^\pm (L_{K^\alpha}^\alpha)_*^r = \mp \alpha (-1)^{srK^\alpha K^\pm} [(((L_{K^\pm}^\pm)_*)^s)_{-\alpha}]^{*(rK^\alpha)}, (L_{K^\alpha}^\alpha)_*^r\}. \quad (26)$$

The Lax-pair representation (26) generates the following equations for the functionals  $u_{k,i}^{(r)}, v_{k,i}^{(r)}$ :

$$\begin{aligned} D_s^+ u_{k,i}^{(r)} &= \sum_{p=1}^k ((-1)^{rpK^++1} u_{p+sK^+,i}^{(s)} u_{k-p,i-p}^{(r)} \\ &+ (-1)^{(k+p)sK^+} u_{k-p,i}^{(r)} u_{p+sK^+,i+p-k+rK^+}^{(s)}), \end{aligned} \quad (27)$$

$$\begin{aligned} D_s^- u_{k,i}^{(r)} &= \sum_{p=0}^{sK^--1} ((-1)^{(sK^-+p)rK^+} v_{p,i}^{(s)} u_{p+k-sK^-,i+p-sK^-}^{(r)} \\ &- (-1)^{(k+p+1)sK^-} u_{p+k-sK^-,i}^{(r)} v_{p,i-p-k+sK^-+rK^+}^{(s)}), \end{aligned} \quad (28)$$

$$\begin{aligned} D_s^+ v_{k,i}^{(r)} &= \sum_{p=0}^{sK^+} ((-1)^{(sK^++p)rK^-} u_{p,i}^{(s)} v_{p+k-sK^+,i-p+sK^+}^{(r)} \\ &- (-1)^{(k+p+1)sK^+} v_{p+k-sK^+,i}^{(r)} u_{p,i+p+k-sK^+-rK^-}^{(s)}), \end{aligned} \quad (29)$$

$$\begin{aligned} D_s^- v_{k,i}^{(r)} &= \sum_{p=0}^k ((-1)^{rpK^-+1} v_{p+sK^-,i}^{(s)} v_{k-p,i+p}^{(r)} \\ &+ (-1)^{(k+p)sK^-} v_{k-p,i}^{(r)} v_{p+sK^-,i+k-p-rK^-}^{(s)}). \end{aligned} \quad (30)$$

It is assumed that in the right-hand side of eqs. (27–30) all the functionals  $u_{k,i}^{(r)}, v_{k,i}^{(r)}$  with  $k < 0$  should be set equal to zero.

Let us demonstrate that all known up to now 2D supersymmetric Toda lattice equations can be derived from the system (27–30).

First, the 2D generalized fermionic Toda lattice equation discussed in [11] can be reproduced from the system of equations (27–30) as a subsystem with additional reduction constraints imposed. In order to see this, let us introduce the notation  $v_{0,i} = d_i, v_{1,i} = \rho_i, u_{1,i} = \gamma_i, u_{2,i} = c_i$  and consider eqs. (28) and (30) at  $K^+ = K^- = 2, r = s = 1$ . One obtains

$$\begin{aligned} D_1^+ d_i &= d_i(c_i - c_{i-2}), & D_1^- \gamma_i &= \rho_i u_{0,i-1} - \rho_{i+2} u_{0,i}, \\ D_1^- c_i &= d_i u_{0,i-2} - d_{i+2} u_{0,i} - \gamma_i \rho_{i+1} - \gamma_{i-1} \rho_i, \\ D_1^+ \rho_i &= \rho_i(c_i - c_{i-1}) + d_{i+1} \gamma_i - d_i \gamma_{i-2}, & D_1^- u_{0,i} &= 0. \end{aligned} \quad (31)$$

It is easy to check that after reduction  $u_{0,i} = 1$  eqs. (31) coincide with the 2D generalized fermionic Toda lattice equations up to time redefinition  $D_1^- \rightarrow -D_1^-$ .

Next, the N=(2|2) supersymmetric Toda lattice equation also belongs to the system (27–30). In order to see that, let us consider eqs. (28) at  $K^+ = K^- = s = k = r = 1$

$$D_1^- u_{1,i} = -u_{0,i-1} v_{0,i} - u_{0,i} v_{0,i+1} \quad (32)$$

and eqs. (29) at  $K^+ = K^- = s = r = 1, k = 0$

$$D_1^+ v_{0,i} = v_{0,i}(u_{1,i} - u_{1,i-1}). \quad (33)$$

Then imposing the constraint  $u_{0,i} = 1$  and eliminating the fields  $u_{1,i}$  from eqs. (32–33) one obtains the  $N=(1|1)$  superfield form of the  $N=(2|2)$  supersymmetric Toda lattice equation [6, 7, 8, 12]

$$D_1^+ D_1^- \ln v_{0,i} = v_{0,i+1} - v_{0,i-1}. \quad (34)$$

Analogously, one can show that the consideration of eqs. (29) and (30) at  $K^+ = 1$ ,  $K^- = 2$ ,  $s = r = 1$  and  $k = 0, 1$  leads to the  $N=(0|1)$  superfield form of the  $N=(0|2)$  supersymmetric Toda lattice equation [12, 9] after imposing the reduction constraints  $u_{0,i} = 1$ ,  $v_{0,2i+1} = 0$ .

We call equations (25) for arbitrary  $(K^+, K^-)$  the 2D fermionic  $(K^+, K^-)$ -Toda lattice hierarchy.

## 5 1D fermionic $(K^+, K^-)$ -Toda lattice hierarchy

In this section, we consider the reduction of the 2D fermionic  $(K^+, K^-)$ -Toda lattice hierarchy for even values of  $(K^+, K^-)$  to the 1D space.

Let  $(K^+, K^-)$  be even numbers. In this case the generalized graded bracket (5) between two  $Z_2$ -even operators turns into the usual commutator and eqs. (25) become

$$D_s^\pm L_{K^\pm}^\alpha = [((L_{K^\pm}^\pm)^s)_\pm, L_{K^\pm}^\alpha]. \quad (35)$$

Following [22] for even  $(K^+, K^-)$  one can impose the reduction constraint on the Lax operators (24) as follows:

$$L_{K^+}^+ + (L_{K^+}^+)^{-1} = L_{K^-}^- + (L_{K^-}^-)^{-1} \equiv L_{K^+, K^-}, \quad (36)$$

which leads to the following explicit form for the reduced Lax operator

$$L_{K^+, K^-} = \sum_{k=0}^{K^+} u_{k,i} e^{(K^+ - k)\partial} + \sum_{k=0}^{K^- - 1} v_{k,i} e^{(k - K^-)\partial} \equiv \sum_{k=0}^{K^+ + K^-} \tilde{u}_{k,i} e^{(K^+ - k)\partial}. \quad (37)$$

Substituting the expressions for the reduced composite Lax operators  $L_{K^\pm}^\pm = L_{K^+, K^-} - (L_{K^\pm}^\pm)^{-1}$  into Lax equations (35) one can see that these equations become equivalent to the single Lax equation on the reduced Lax operator

$$D_s L_{K^+, K^-} = [((L_{K^+, K^-})^s)_+, L_{K^+, K^-}] \quad (38)$$

with  $D_s^+ = -D_s^- = D_s$ . As a consequence of eq. (38) we have

$$D_s (L_{K^+, K^-})^r = [((L_{K^+, K^-})^s)_+, (L_{K^+, K^-})^r].$$

At the reduction  $u_{0,i} = 1$  the 1D  $(2, 2)$ -TL hierarchy becomes that studied in detail in [11]. In this case, the representation (38) with the Lax operator

$$L_{2,2} = e^{2\partial} + \gamma_i e^\partial + c_i + \rho_i e^{-\partial} + d_i e^{-2\partial}$$

gives the following first flow:

$$\begin{aligned}
D_1 d_i &= d_i(c_i - c_{i-2}), \\
D_1 \rho_i &= \rho_i(c_i - c_{i-1}) + d_{i+1}\gamma_i - d_i\gamma_{i-2}, \\
D_1 \gamma_i &= \rho_{i+2} - \rho_i, \\
D_1 c_i &= d_{i+2} - d_i + \gamma_i \rho_{i+1} + \gamma_{i-1} \rho_i.
\end{aligned}$$

These are the 1D generalized fermionic Toda lattice equations [11] which possess the  $N = 4$  supersymmetry.

## 6 Bi-Hamiltonian structure of 1D fermionic $(K^+, K^-)$ -TL hierarchy

In this section, we apply the R-matrix approach to build the bi-Hamiltonian structure of the 1D fermionic  $(K^+, K^-)$ -TL hierarchy and perform its Dirac reduction.

The space of operators  $\mathbb{O}_{K^+}^+$  (4) can obviously be split into the vector space direct sum,  $\mathbb{O}_{K^+} = (\mathbb{O}_{K^+})_+ \dot{+} (\mathbb{O}_{K^+})_-$ . The  $R$ -matrix arising from this splitting

$$R = P_+ - P_-, \quad R(\mathbb{O}_{K^+}) = (\mathbb{O}_{K^+})_+ - (\mathbb{O}_{K^+})_-$$

obviously solves the graded modified Yang-Baxter equation (19) at  $\alpha = 1$ . This  $R$ -matrix is not graded antisymmetric,  $R \neq -R^\dagger$ ; however, its graded antisymmetric part  $A = 1/2(R - R^\dagger)$  as well as the  $R$ -matrix itself satisfy the graded modified Yang-Baxter equation (19). According to the general Theorem of Section 3 this means that there exist two Poisson structures on  $\mathfrak{g}^\dagger = \mathfrak{g}$ . Substituting the general form of operators  $L_{K^+}^+$  (24) and

$$\nabla u_{n,\xi} = e^{(n-K^+)\partial} (-1)^i \delta_{i,\xi}$$

into (20) and (21) one can find the explicit form of the first and second Poisson brackets, respectively,

$$\begin{aligned}
\{u_{n,i}, u_{m,j}\}_1 &= (-1)^j (\delta_{n,K^+}^- + \delta_{m,K^+}^- - 1) (u_{n+m-K^+,i} \delta_{i,j+n-K^+} \\
&\quad - (-1)^{(m+K^+)(n+K^++1)} u_{n+m-K^+,j} \delta_{i,j-m+K^+})
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
\{u_{n,i}, u_{m,j}\}_2 &= -(-1)^j \frac{1}{2} \left[ u_{n,i} u_{m,j} (\delta_{i,j+n-K^+} - (-1)^m \delta_{i,j-m+K^+}) \right. \\
&\quad + \sum_{k=0}^{n+m} (\delta_{m,k}^+ - \delta_{m,k}^-) \left( (-1)^{mk} u_{n+m-k,i} u_{k,j} \delta_{i,j+n-k} \right. \\
&\quad \left. \left. - (-1)^{m(n+k+1)} u_{k,i} u_{n+m-k,j} \delta_{i,j-m+k} \right) \right],
\end{aligned} \tag{40}$$

where

$$\delta_{n,m}^+ = \begin{cases} 1, & \text{if } n > m \\ 0, & \text{if } n \leq m \end{cases}, \quad \delta_{n,m}^- = \begin{cases} 1, & \text{if } n < m \\ 0, & \text{if } n \geq m. \end{cases}$$

Let us remind that the second Poisson brackets are defined for even values of  $K^+$  only.

Our next goal is to perform the reduction of Poisson brackets (39–40) for the functions parameterizing the operators  $L_{K^+}^+$  (24) to the Poisson brackets corresponding to the reduced operators (37)

$$L_{K^+, K^-}^{red} = e^{K^+ \partial} + \sum_{k=1}^{K^+ + K^-} u_{k,i} e^{(K^+ - k) \partial},$$

where  $K^+, K^-$  are even numbers. Therefore, one needs to modify the Poisson brackets (39–40) according to the reduction constraints

$$\left. \begin{aligned} u_{k,i} &= 0, & k > K^+ + K^-, \\ u_{0,i} &= 1 \end{aligned} \right\} \text{ for any } i. \quad (41)$$

We apply these reduction constraints in two steps. First, we note that for the first constraint in (41) the reduction simply amounts to imposing constraint  $u_{k,i} = 0$  ( $k > K^+ + K^-$ ) due to the observation that

$$\left. \{u_{n,i}, u_{m,j}\}_p \right|_{\substack{u_{k,i}=0, \\ k > K^+ + K^-}} = 0, \quad 0 \leq n \leq K^+ + K^-, \quad m > K^+ + K^-, \quad p = 1, 2. \quad (42)$$

For the first Poisson brackets (39) relation (42) is obvious. One can derive eq. (42) for the second Poisson brackets (40) if one divides the sum in (40) into three pieces

$$\sum_{k=0}^{n+m} = \sum_{k=0}^{\max(0, n+m-K^+-K^- - 1)} + \sum_{k=\max(1, n+m-K^+-K^-)}^{\min(n+m-1, K^++K^-)} + \sum_{k=\min(n+m, K^++K^-+1)}^{n+m}. \quad (43)$$

Now it is easy to verify that the second sum in the r.h.s. of eq. (43) is the only sum which could give a nonzero contribution to eq. (42), but it is equal to zero if  $0 \leq n \leq K^+ + K^-$ ,  $m > K^+ + K^-$ .

Now let us consider the second reduction constraint in eq. (41),  $u_{0,i} = 1$ . Following the standard Dirac reduction prescription we obtain for the Dirac brackets

$$\{u_{n,i}, u_{m,j}\}_p^{red} = \left( \{u_{n,i}, u_{m,j}\}_p \right) \Big|_{u_{0,i}=1} - \Delta_p(u_{n,i}, u_{m,j}), \quad p = 1, 2$$

with the correction term

$$\Delta_p(u_{n,i}, u_{m,j}) = \left( \sum_{i', j'} \{u_{n,i}, u_{0,i'}\}_p \{u_{0,i'}, u_{0,j'}\}_p^{-1} \{u_{0,j'}, u_{m,j}\}_p \right) \Big|_{u_{0,i}=1}.$$

In the case of the first Poisson brackets  $\{u_{0,i}, u_{m,j}\}_1 = 0$  for any  $m$ . Thus, one can conclude that the first Poisson brackets (39) are not modified, and they can simply be restricted by imposing the constraints (41).

Before investigating the second Poisson brackets (40) we supply the fields  $u_{n,i}$  and  $v_{n,i}$  with the boundary conditions

$$\lim_{i \rightarrow \pm\infty} u_{n,i} = \lim_{i \rightarrow \pm\infty} v_{n,i} = 0, \quad n \neq 0 \quad (44)$$

and introduce a new notation  $\delta_{i,j-n} := (\Lambda^n)_{i,j}$  which is useful in what follows. One can verify that in the new notation the multiplication of matrices results in adding powers of the operators  $\Lambda$ :

$$\delta_{i,j'+n} \delta_{j',j+m} := \Lambda_{i,j'}^{-n} \Lambda_{j',j}^{-m} = \Lambda_{i,j}^{-m-n}.$$

Then, one can represent the correction term for the reduced second Poisson brackets as follows:

$$\begin{aligned} \triangle_2(u_{n,i}, u_{m,j}) &= 1/2(-1)^j u_{n,i} [(1 - \Lambda^{K^+-n})(1 + \Lambda^{-K^+}) \\ &\quad (\Lambda^{K^+} - \Lambda^{-K^+})^{-1} (1 + \Lambda^{K^+})(1 - (-1)^m \Lambda^{m-K^+})]_{i,j} u_{m,j}. \end{aligned}$$

In general, the reduced second Poisson brackets are nonlocal since the inverse matrix

$$(\Lambda^{2\nu} - \Lambda^{-2\nu})^{-1} = (1 + \Lambda^{2\nu})^{-1} (1 - \Lambda^{-\nu})^{-1} (1 + \Lambda^{-\nu})^{-1},$$

being considered as an operator acting in the space of functionals with boundary conditions (44) can be expressed via infinite sums

$$\begin{aligned} (1 - \Lambda^{-\nu})^{-1} &= \lambda_1 \sum_{k=0}^{\infty} \Lambda^{-k\nu} - (1 - \lambda_1) \sum_{k=1}^{\infty} \Lambda^{k\nu}, \\ (1 + \Lambda^{-\nu})^{-1} &= \lambda_2 \sum_{k=0}^{\infty} (-1)^k \Lambda^{-k\nu} + (1 - \lambda_2) \sum_{k=1}^{\infty} (-1)^k \Lambda^{k\nu}, \end{aligned}$$

where  $\lambda_1$  and  $\lambda_2$  are arbitrary parameters. However, in the particular case  $K^+ = 2$  the Dirac bracket becomes local, since the nonlocality is eliminated due to the contraction of the matrix with its inverse matrix. Indeed, one has

$$\begin{aligned} 1 - \Lambda^\nu &= (1 - \Lambda^{-1})(\delta_{\nu,0}^- \sum_{k=\nu+1}^0 \Lambda^k - \delta_{\nu,0}^+ \sum_{k=1}^{\nu} \Lambda^k), \\ 1 - (-1)^\nu \Lambda^\nu &= (1 + \Lambda^{-1})(\delta_{\nu,0}^- \sum_{k=\nu+1}^0 (-1)^k \Lambda^k - \delta_{\nu,0}^+ \sum_{k=1}^{\nu} (-1)^k \Lambda^k) \end{aligned}$$

and for  $K^+ = 2$  the Dirac brackets are local

$$\begin{aligned} \triangle_2(u_{n,i}, u_{m,j}) &= 1/2(-1)^j u_{n,i} [(1 + \Lambda^{-2}) \\ &\quad (\delta_{n,2}^+ \sum_{k=3-n}^0 \Lambda^k - \delta_{n,2}^- \sum_{k=1}^{2-n} \Lambda^k)(\delta_{m,2}^- \sum_{s=m-1}^0 (-1)^s \Lambda^s - \delta_{m,2}^+ \sum_{s=1}^{m-2} (-1)^s \Lambda^s)]_{i,j} u_{m,j}. \end{aligned} \quad (45)$$

As an example, we finish this section discussing the explicit form of the second Hamiltonian structure of the 1D (2, 2)-Toda lattice hierarchy. The Lax operator defining this hierarchy is parameterized as follows:

$$L_{2,2} = u_i e^{2\partial} + \gamma_i e^{\partial} + c_i + \rho_i e^{-\partial} + d_i e^{-2\partial}.$$

Using eqs. (39), (40) and (45) one can derive the corresponding first Hamiltonian structure

$$\begin{aligned} \{d_i, c_j\}_1 &= (-1)^j d_i (\delta_{i,j} - \delta_{i,j+2}), \\ \{c_i, \rho_j\}_1 &= -(-1)^j \rho_j (\delta_{i,j-1} + \delta_{i,j}), \\ \{\rho_i, \rho_j\}_1 &= (-1)^j (d_i \delta_{i,j-1} - d_j \delta_{i,j+1}), \\ \{\gamma_i, \gamma_j\}_1 &= (-1)^j (\delta_{i,j-1} - \delta_{i,j+1}) \end{aligned} \quad (46)$$

and the second Hamiltonian structure

$$\begin{aligned} \{d_i, d_j\}_2 &= 1/2 (-1)^j d_i d_j (1 + \Delta) (\delta_{i,j-2} - \delta_{i,j+2}), \\ \{d_i, \rho_j\}_2 &= 1/2 (-1)^j d_i \rho_j ((1 - \Delta) (\delta_{i,j} + \delta_{i,j+1}) - (1 + \Delta) (\delta_{i,j-1} + \delta_{i,j+2})), \\ \{d_i, c_j\}_2 &= (-1)^j d_i c_j (\delta_{i,j} - \delta_{i,j+2}), \\ \{d_i, \gamma_j\}_2 &= 1/2 (-1)^j d_i \gamma_j ((1 - \Delta) (\delta_{i,j} + \delta_{i,j+3}) - (1 + \Delta) (\delta_{i,j+1} + \delta_{i,j+2})), \\ \{d_i, u_j\}_2 &= 1/2 (-1)^j d_i u_j (1 - \Delta) (\delta_{i,j} - \delta_{i,j-4}), \\ \{c_i, c_j\}_2 &= (-1)^j (u_i d_j \delta_{i,j-2} - u_j d_i \delta_{i,j+2} + \gamma_i \rho_j \delta_{i,j-1} + \gamma_j \rho_i \delta_{i,j+1}), \\ \{c_i, \rho_j\}_2 &= (-1)^j (d_i \gamma_j \delta_{i,j+1} + d_j \gamma_i \delta_{i,j-2} - c_i \rho_j (\delta_{i,j} + \delta_{i,j-1})), \\ \{c_i, \gamma_j\}_2 &= -(-1)^j (u_i \rho_j \delta_{i,j-1} + u_j \rho_i \delta_{i,j+2}), \\ \{\rho_i, \rho_j\}_2 &= (-1)^j (c_i d_j \delta_{i,j-1} - c_j d_i \delta_{i,j+1} - 1/2 \rho_i \rho_j (1 + \Delta) (\delta_{i,j+1} + \delta_{i,j-1})), \\ \{\rho_i, \gamma_j\}_2 &= (-1)^j (u_i d_j \delta_{i,j-1} - u_j d_i \delta_{i,j+3} - \rho_i \gamma_j (\delta_{i,j+1} - 1/2 (1 - \Delta) (\delta_{i,j} + \delta_{i,j-2}))), \\ \{\rho_i, u_j\}_2 &= 1/2 (-1)^j \rho_i u_j (1 - \Delta) (\delta_{i,j} - \delta_{i,j+1} + \delta_{i,j+2} - \delta_{i,j+3}), \\ \{\gamma_i, \gamma_j\}_2 &= (-1)^j (u_i c_j \delta_{i,j-1} - u_j c_i \delta_{i,j+1} - 1/2 \gamma_i \gamma_j (1 - \Delta) (\delta_{i,j+1} + \delta_{i,j-1})), \\ \{\gamma_i, u_j\}_2 &= 1/2 (-1)^j \gamma_i u_j (1 - \Delta) (\delta_{i,j} - \delta_{i,j+1} + \delta_{i,j+2} - \delta_{i,j-1}), \\ \{u_i, u_j\}_2 &= 1/2 (-1)^j u_i u_j (1 - \Delta) (\delta_{i,j+2} - \delta_{i,j-2}), \end{aligned} \quad (47)$$

where only nonzero brackets are written down. Here we have introduced the parameter  $\Delta$  which for the unreduced brackets is equal to zero,  $\Delta = 0$ , and for the Dirac reduced brackets with the reduction constraint  $u_i = 1$  is equal to one,  $\Delta = 1$ . In the latter case, algebras (46) and (47) reproduce, respectively, the first and second Hamiltonian structures of the 1D generalized fermionic Toda lattice hierarchy found in [11] by a heuristic approach.

## 7 Bi-Hamiltonian structure of 2D fermionic $(K^+, K^-)$ -TL hierarchy

In this section, we construct the bi-Hamiltonian structure of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy. This hierarchy is associated with two Lax operators (24) belonging to the operator

space (4). Following [13] we consider the associative algebra on the space of the direct sum of two difference operators

$$\mathbf{g} := \mathbb{O}_{K+}^+ \oplus \mathbb{O}_{K-}^- . \quad (48)$$

However, in contrast to the case of pure bosonic 2D TL hierarchy, the difference operators in the direct sum (48) can be of both opposite and equal diagonal  $Z_2$ -parity. It turns out that the Poisson brackets can correctly be defined only for the latter case. In what follows we restrict ourselves to the case when both operators in  $\mathbf{g}$  (48) have the same diagonal parity.

We denote  $(x^+, x^-)$  elements of such algebra  $\mathbf{g} = \mathbf{g}^\dagger$  with the product

$$(x_1^+, x_1^-) \cdot (x_2^+, x_2^-) = (x_1^+ x_2^+, x_1^- x_2^-), \quad (49)$$

and define the inner product as follows:

$$\langle x^+, x^- \rangle := \text{str}(x^+ + x^-), \quad (50)$$

where  $x^+ \in \mathbb{O}_{K+}^+$ ,  $x^- \in \mathbb{O}_{K-}^-$ . Using this definition we set the Poisson bracket to be

$$\{f_1, f_2\} = \langle (\mathbb{O}_{K+}^+, \mathbb{O}_{K-}^-), [\nabla f_1, \nabla f_2]^\oplus \rangle, \quad (51)$$

where

$$[\nabla f_1, \nabla f_2]^\oplus := ([\nabla f_1^+, (\nabla f_2^+)^{*(d_{\nabla f_1^+})}], [\nabla f_1^-, (\nabla f_2^-)^{*(d_{\nabla f_1^-})}]),$$

$f_k$  are functionals on  $\mathbf{g}$  (48), and  $\nabla f_k[(\mathbb{O}_{K+}^+, \mathbb{O}_{K-}^-)] = (\nabla f_k^+, \nabla f_k^-)$  are their gradients which can be found from the definition

$$\begin{aligned} \left. \frac{\partial f_k[(\mathbb{O}_{K+}^+, \mathbb{O}_{K-}^-) + \epsilon(\delta\mathbb{O}_{K+}^+, \delta\mathbb{O}_{K-}^-)]}{\partial \epsilon} \right|_{\epsilon=0} &= \langle (\delta\mathbb{O}_{K+}^+, \delta\mathbb{O}_{K-}^-), (\nabla f_k^+, \nabla f_k^-) \rangle \\ &= \langle \delta\mathbb{O}_{K+}^+, \nabla f_k^+ \rangle + \langle \delta\mathbb{O}_{K-}^-, \nabla f_k^- \rangle . \end{aligned}$$

In order to obtain nontrivial Hamiltonian dynamics, one needs to modify the bracket (51) applying the  $R$ -matrix

$$[\nabla f_1, \nabla f_2]^\oplus \longrightarrow [\nabla f_1, \nabla f_2]_R^\oplus = [R(\nabla f_1), \nabla f_2]^\oplus + [\nabla f_1, R(\nabla f_2)]^\oplus.$$

The  $R$ -matrix acts on the space (48) in the nontrivial way and mixes up the elements from two subalgebras in the direct sum with each other

$$R(x^+, x^-) = (x_+^+ - x_-^+ + 2x_-^-, x_-^- - x_+^- + 2x_+^+) \quad (52)$$

which is a crucial point of the  $R$ -matrix approach in the two-dimensional case [13]. This  $R$ -matrix allows one to find two compatible Poisson structures and rewrite the Lax-pair representation (25) in the Hamiltonian form.

By construction the  $R$ -matrix (52) satisfies the graded modified Yang-Baxter equation

$$R([(x^+, x^-), (y^+, y^-)]_R) - [R(x^+, x^-), R(y^+, y^-)] = \alpha[(x^+, x^-), (y^+, y^-)] \quad (53)$$



with  $\alpha = 1$ . In order to show this, we simply repeat the arguments of [13] representing the  $R$ -matrix as the difference  $R = \Pi - \bar{\Pi}$  of two projection operators

$$\Pi(x^+, x^-) = (x_+^+ + x_-^-, x_+^+ + x_-^-), \quad \bar{\Pi}(x^+, x^-) = (x_-^+ - x_-^-, x_+^- - x_+^+)$$

$$\Pi(x^+, x^-) + \bar{\Pi}(x^+, x^-) = (x^+, x^-), \quad \Pi^2 = \Pi, \quad \bar{\Pi}^2 = \bar{\Pi}, \quad \bar{\Pi}\Pi = \Pi\bar{\Pi} = 0.$$

Therefore, the  $R$ -matrix (52) provides the splitting of the algebra  $\mathfrak{g}$  and solves the modified graded Yang-Baxter equation (53).

The two-dimensional  $R$ -matrix is not graded antisymmetric, its adjoint counterpart  $R^\dagger$  looks like

$$R^\dagger(x^+, x^-) = (x_{\leq 0}^+ - x_{> 0}^+ + 2x_{\leq 0}^-, x_{> 0}^- - x_{\leq 0}^- + 2x_{> 0}^+) = \Pi^\dagger - \bar{\Pi}^\dagger,$$

where the dual projections are

$$\begin{aligned} \Pi^\dagger(x^+, x^-) &= (x_{\leq 0}^+ + x_{\leq 0}^-, x_{> 0}^+ + x_{> 0}^-), \\ \bar{\Pi}^\dagger(x^+, x^-) &= (x_{> 0}^+ - x_{\leq 0}^-, x_{\leq 0}^- - x_{> 0}^+). \end{aligned}$$

The direct verification by substitution in (53) shows that the graded antisymmetric part

$$1/2(R(x^+, x^-) - R^\dagger(x^+, x^-)) = (x_{> 0}^+ - x_{< 0}^+ - x_0^-, x_{< 0}^- - x_{> 0}^- + x_0^+)$$

also satisfies the graded modified Yang-Baxter equation (53). Therefore, by Theorem of Section 3 there exist two Poisson structures on  $\mathfrak{g}$  (48).

Using eqs. (20–21), (49–50), (52) and cyclic permutations inside the supertrace (9) we obtain the following general form of the first and second Poisson brackets:

$$\begin{aligned} \{f, g\}_i &= \langle P_i^+(\nabla g^+, \nabla g^-), (\nabla f^+)^{*(d_{\nabla g})} \rangle \\ &+ \langle P_i^-(\nabla g^+, \nabla g^-), (\nabla f^-)^{*(d_{\nabla g})} \rangle, \quad i = 1, 2, \end{aligned} \quad (54)$$

where  $d_{\nabla g} := d_{\nabla g^+} = d_{\nabla g^-}$ . The Poisson tensors in eq. (54) are found for any values of  $(K^+, K^-)$  for the first Hamiltonian structure

$$\begin{aligned} P_1^+(\nabla g^+, \nabla g^-) &= [(\nabla g_-^- - \nabla g_-^+)^{*(K^+)}, (L_{K^+}^+)^{*(d_{\nabla g})}] \\ &- ([L_{K^+}^+, \nabla g^+] + [L_{K^-}^-, \nabla g^-])_{\leq 0}, \\ P_1^-(\nabla g^+, \nabla g^-) &= [(\nabla g_+^+ - \nabla g_+^-)^{*(K^-)}, (L_{K^-}^-)^{*(d_{\nabla g})}] \\ &- ([L_{K^+}^+, \nabla g^+] + [L_{K^-}^-, \nabla g^-])_{> 0}, \end{aligned}$$

while for the second Hamiltonian structure we constructed the explicit expression of the Poisson tensors for even values of  $(K^+, K^-)$  only

$$\begin{aligned} P_2^+(\nabla g^+, \nabla g^-) &= 1/2 \left( [(\nabla g^-(L_{K^-}^-)^{*(d_g)} + L_{K^-}^- \nabla g^- \right. \\ &- \left. \nabla g^+(L_{K^+}^+)^{*(d_g)} - L_{K^+}^+ \nabla g^+)^-, (L_{K^+}^+)^{*(d_g)} \right] \end{aligned}$$

$$\begin{aligned}
& - L_{K+}^+ ([L_{K+}^+, \nabla g^+] + [L_{K-}^-, \nabla g^-])_{\leq 0} \\
& - ([L_{K+}^+, \nabla g^+] + [L_{K-}^-, \nabla g^-])_{\leq 0} (L_{K+}^+)^{*(d_g)}, \\
P_2^-(\nabla g^+, \nabla g^-) &= 1/2 \left( [(\nabla g^+ (L_{K+}^+)^{*(d_g)} + L_{K+}^+ \nabla g^+ \right. \\
& - \nabla g^- (L_{K-}^-)^{*(d_g)} - L_{K-}^- \nabla g^-)_{+}, (L_{K-}^-)^{*(d_g)} \} \\
& - L_{K-}^- ([L_{K+}^+, \nabla g^+] + [L_{K-}^-, \nabla g^-])_{> 0} \\
& - ([L_{K+}^+, \nabla g^+] + [L_{K-}^-, \nabla g^-])_{> 0} (L_{K-}^-)^{*(d_g)} \}.
\end{aligned}$$

The Poisson brackets for the functions  $u_{n,i}$  and  $v_{n,i}$  parameterizing the Lax operators (24) can explicitly be derived from (54) if one takes into account that

$$\begin{aligned}
\nabla u_{n,\xi} &\equiv (\nabla u_{n,\xi}^+, \nabla u_{n,\xi}^-) = (e^{(n-K^+)\partial} (-1)^i \delta_{i,\xi}, 0), \\
\nabla v_{n,\xi} &\equiv (\nabla v_{n,\xi}^+, \nabla v_{n,\xi}^-) = (0, e^{(K^- - n)\partial} (-1)^i \delta_{i,\xi}).
\end{aligned}$$

In such a way one can obtain the following expressions:

$$\begin{aligned}
\{u_{n,i}, u_{m,j}\}_1 &= (-1)^j (\delta_{n,K^+}^- + \delta_{m,K^+}^- - 1) \left( u_{n+m-K^+,i} \delta_{i,j+n-K^+} \right. \\
&\quad \left. - (-1)^{(m+K^+)(n+K^++1)} u_{n+m-K^+,j} \delta_{i,j-m+K^+} \right), \\
\{u_{n,i}, v_{m,j}\}_1 &= (-1)^j \left[ \delta_{m,K^-}^+ \left( (-1)^{(m+K^-)(n+K^++1)} u_{n-m+K^-,j} \delta_{i,j+m-K^-} \right. \right. \\
&\quad \left. - u_{n-m+K^-,i} \delta_{i,j+n-K^+} \right) + (\delta_{n,K^+}^- - 1) \left( v_{m-n+K^+,i} \delta_{i,j+n-K^+} \right. \\
&\quad \left. - (-1)^{(m+K^-)(n+K^++1)} v_{m-n+K^+,j} \delta_{i,j+m-K^-} \right) \Big], \\
\{v_{n,i}, v_{m,j}\}_1 &= (-1)^j (1 - \delta_{n,K^-}^+ - \delta_{m,K^-}^+) \left( v_{n+m-K^-,i} \delta_{i,j-n+K^-} \right. \\
&\quad \left. - (-1)^{(m+K^-)(n+K^-+1)} u_{n+m-K^-,j} \delta_{i,j+m-K^-} \right)
\end{aligned} \tag{55}$$

for the first Hamiltonian structure and

$$\begin{aligned}
\{u_{n,i}, u_{m,j}\}_2 &= -(-1)^j \frac{1}{2} \left[ u_{n,i} u_{m,j} (\delta_{i,j+n-K^+} - (-1)^m \delta_{i,j-m+K^+}) \right. \\
&\quad + \sum_{k=0}^{n+m} (\delta_{m,k}^+ - \delta_{m,k}^-) \left( (-1)^{mk} u_{n+m-k,i} u_{k,j} \delta_{i,j+n-k} \right. \\
&\quad \left. - (-1)^{m(n+k+1)} u_{k,i} u_{n+m-k,j} \delta_{i,j-m+k} \right) \Big], \\
\{u_{n,i}, v_{m,j}\}_2 &= -(-1)^j \frac{1}{2} \left[ u_{n,i} v_{m,j} \left( \delta_{i,j} + \delta_{i,j+n-K^+} \right. \right. \\
&\quad \left. - (-1)^m (\delta_{i,j+m-K^-} + \delta_{i,j+n+m-K^+-K^-}) \right) \\
&\quad + 2 \sum_{k=\max(0,m-n)}^{m-1} \left( u_{n-m+k,i} v_{k,j+m-k} \delta_{i,j+n-K^+} \right.
\end{aligned}$$

$$\begin{aligned}
& - (-1)^{(n+1)m} v_{k,j} u_{n-m+k,i+k-m} \delta_{i,j+m+K^-} \Big) \Big], \\
\{v_{n,i}, v_{m,j}\}_2 &= (-1)^j \frac{1}{2} \Big[ v_{n,i} v_{m,j} (\delta_{i,j-n+K^-} - (-1)^m \delta_{i,j+m-K^-}) \\
& - \sum_{k=0}^{n+m} (\delta_{m,k}^+ - \delta_{m,k}^-) \Big( (-1)^{mk} v_{n+m-k,i} v_{k,j} \delta_{i,j-n+k} \\
& - (-1)^{m(n+k+1)} v_{k,i} v_{n+m-k,j} \delta_{i,j+m-k} \Big) \Big] \quad (56)
\end{aligned}$$

for the second Hamiltonian structure; the latter is valid for even values of  $(K^+, K^-)$  only.

The reduction, according the reduction constraint  $u_{0,i} = 1$ , does not require any correction terms for the first Hamiltonian structure (55) because  $\{u_{0,i}, u_{n,j}\} = \{u_{0,i}, v_{n,j}\} = 0$ . For the second Hamiltonian structure the correction terms are

$$\begin{aligned}
\Delta_2(u_{n,i}, u_{m,j}) &= 1/2 (-1)^j u_{n,i} (1 - \Lambda^{K^+ - n}) (1 + \Lambda^{-K^+}) \\
&\quad (\Lambda^{K^+} - \Lambda^{-K^+})^{-1} (1 + \Lambda^{K^+}) (1 - (-1)^m \Lambda^{m-K^+}) u_{m,j}, \\
\Delta_2(u_{n,i}, v_{m,j}) &= 1/2 (-1)^j u_{n,i} (1 - \Lambda^{K^+ - n}) (1 + \Lambda^{-K^+}) \\
&\quad (\Lambda^{K^+} - \Lambda^{-K^+})^{-1} (1 + \Lambda^{K^+}) (1 - (-1)^m \Lambda^{K^- - m}) v_{m,j}, \\
\Delta_2(v_{n,i}, v_{m,j}) &= 1/2 (-1)^j v_{n,i} (1 - \Lambda^{n-K^-}) (1 + \Lambda^{-K^+}) \\
&\quad (\Lambda^{K^+} - \Lambda^{-K^+})^{-1} (1 + \Lambda^{K^+}) (1 - (-1)^m \Lambda^{K^- - m}) v_{m,j} \quad (57)
\end{aligned}$$

and they are nonlocal due to the presence of  $(\Lambda^{K^+} - \Lambda^{-K^+})^{-1}$  in the r.h.s. of eq. (57). However, there are unique values of  $(K^+, K^-)$  when nonlocal terms are eliminated. Indeed, for  $K^+ = K^- = 2$  eqs. (57) become local

$$\begin{aligned}
\Delta_2(u_{n,i}, u_{m,j}) &= 1/2 (-1)^j u_{n,i} (1 + \Lambda^{-2}) \\
&\quad (\delta_{n,2}^+ \sum_{k=3-n}^0 \Lambda^k - \delta_{n,2}^- \sum_{k=1}^{2-n} \Lambda^k) (\delta_{m,2}^- \sum_{s=m-1}^0 (-1)^s \Lambda^s - \delta_{m,2}^+ \sum_{s=1}^{m-2} (-1)^s \Lambda^s) u_{m,j}, \\
\Delta_2(u_{n,i}, v_{m,j}) &= 1/2 (-1)^j u_{n,i} (1 + \Lambda^{-2}) \\
&\quad (\delta_{n,2}^+ \sum_{k=3-n}^0 \Lambda^k - \delta_{n,2}^- \sum_{k=1}^{2-n} \Lambda^k) (\delta_{m,2}^+ \sum_{s=3-m}^0 (-1)^s \Lambda^s - \delta_{m,2}^- \sum_{s=1}^{2-m} (-1)^s \Lambda^s) v_{m,j}, \\
\Delta_2(v_{n,i}, v_{m,j}) &= 1/2 (-1)^j v_{n,i} (1 + \Lambda^{-2}) \\
&\quad (\delta_{n,2}^- \sum_{k=n-1}^0 \Lambda^k - \delta_{n,2}^+ \sum_{k=1}^{n-2} \Lambda^k) (\delta_{m,2}^+ \sum_{s=3-m}^0 (-1)^s \Lambda^s - \delta_{m,2}^- \sum_{s=1}^{2-m} (-1)^s \Lambda^s) v_{m,j}.
\end{aligned}$$

The Hamiltonian structures thus obtained possess the properties (12-14) with  $d_{\mathbb{O}} = d_{L_{K^+}^+} = d_{L_{K^-}^-}$ . Using them one can rewrite flows (27-30) for even values of  $(K^+, K^-)$  in the bi-Hamiltonian form

$$D_s^\pm \begin{pmatrix} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{pmatrix} = \left\{ \begin{pmatrix} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{pmatrix}, H_{s+1}^\pm \right\}_1 = \left\{ \begin{pmatrix} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{pmatrix}, H_s^\pm \right\}_2,$$

with Hamiltonians

$$\begin{aligned} H_s^+ &= \frac{1}{s} \text{str}(L_{K^+}^+)_*^s = \frac{1}{s} \sum_{i=-\infty}^{\infty} (-1)^i u_{sK^+,i}^{(s)}, \\ H_s^- &= \frac{1}{s} \text{str}(L_{K^-}^-)_*^s = \frac{1}{s} \sum_{i=-\infty}^{\infty} (-1)^i v_{sK^-,i}^{(s)}. \end{aligned} \quad (58)$$

For odd values of  $(K^+, K^-)$  one can reproduce the bosonic flows of (27–30) only. In this case eqs. (58), due to relation (17), give only fermionic nonzero Hamiltonians using which the bosonic flows can be generated via odd first Hamiltonian structure (55)

$$D_{2s}^{\pm} \left( \begin{pmatrix} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{pmatrix} \right) = \left\{ \begin{pmatrix} u_{n,i}^{(r)} \\ v_{n,i}^{(r)} \end{pmatrix}, H_{2s+1}^{\pm} \right\}_1.$$

One remark is in order. In Sec. 6, we derived the bi-Hamiltonian structure for the 1D fermionic  $(K^+, K^-)$ -TL hierarchy in the  $R$ -matrix approach applying the  $R$ -matrix formalism developed in Sec. 3. However, the 1D fermionic  $(K^+, K^-)$ -TL hierarchy is obtained as a reduction of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy with the reduction constraint (36). Therefore, this reduction constraint can be carried over into Hamiltonian structures and the bi-Hamiltonian structure for the 1D fermionic  $(K^+, K^-)$ -TL hierarchy can equivalently be derived from that of the 2D hierarchy just by reduction with the corresponding constraint. Actually, this reduction amounts to the extraction of subalgebras in the Hamiltonian structures for the 2D fermionic  $(K^+, K^-)$ -TL hierarchy. Indeed, one can verify that the fields  $u_{n,i}$  ( $0 \leq n \leq K^+$ ) and  $v_{n,i}$  ( $0 \leq n \leq K^- - 1$ ) form subalgebras for even values of  $(K^+, K^-)$  in both the first (55) and the second (56) Hamiltonian structures and these subalgebras are the first (39) and the second (40) Hamiltonian structures, respectively, if one redefines the fields as follows:

$$\begin{aligned} u_{n,i} &= u_{n,i}, & 0 \leq n \leq K^+ \\ u_{K^++K^--n,i} &= -v_{n,i}, & 0 \leq n \leq K^- - 1. \end{aligned}$$

## 8 Conclusion

In this paper, we have generalized the  $R$ -matrix method to the case of  $Z_2$ -graded operators with an involution and found that there exist two Poisson bracket structures. The first Poisson bracket is defined for both odd and even operators with  $Z_2$ -grading while the second one is found for even operators only. It was shown that properties of the Poisson brackets were provided by the properties of the generalized graded bracket. We have deduced the operator form of the graded modified Yang-Baxter equation and demonstrated that for the class of graded antisymmetric  $R$ -matrices it was equivalent to the tensor form of the graded classical Yang-Baxter equation. Then we have proposed the Lax-pair representation in terms of the generalized graded bracket of the new 2D fermionic  $(K^+, K^-)$ -Toda lattice hierarchy and demonstrated that this hierarchy included all known up to now 2D supersymmetric TL equations as subsystems.

Next we have considered the reduction of this hierarchy to the 1D space and reproduced the 1D generalized fermionic TL equations [11]. Finally, we have applied the developed R-matrix formalism to derive the bi-Hamiltonian structure of the 1D and 2D fermionic  $(K^+, K^-)$ -TL hierarchies. For even values of  $(K^+, K^-)$  both even first and second Hamiltonian structures were obtained and for this case all the flows of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy can be rewritten in a bi-Hamiltonian form. For odd values of  $(K^+, K^-)$  odd first Hamiltonian structure was found and for this case only bosonic flows of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy can be represented in a Hamiltonian form using fermionic Hamiltonians.

Thus, the problem of Hamiltonian description of the fermionic flows of the 2D fermionic  $(K^+, K^-)$ -TL hierarchy is still open. Other problems yet to be answered are the construction of the second Hamiltonian structure (if any) for odd Lax operators and of the Hamiltonian structures (if any) for Lax operators  $L_K^+$  and  $L_M^-$  of opposite  $Z_2$ -parities. All these questions are a subject for future investigations.

**Acknowledgments.** We would like to thank A.P. Isaev, P.P. Kulish, and A.A. Vladimirov for useful discussions. This work was partially supported by RFBR-DFG Grant No. 04-02-04002, DFG Grant 436 RUS 113/669-2, the NATO Grant PST.GLG.980302, and by the Heisenberg-Landau program.

## Appendix A

Here we show that the graded modified Yang-Baxter equation (19) for the case of graded antisymmetric operators  $R$  is equivalent to the tensor form of the graded classical Yang-Baxter equation introduced in the pioneer paper [15]

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0. \quad (\text{A.1})$$

Let  $\mathcal{G}$  be a superalgebra with the generators  $e_\mu$  ( $\mu = 1, \dots, n + m$ ), the structure constants  $C_{\mu\nu}^\rho$  and the graded Lie bracket

$$\{e_\mu, e_\nu\} = e_\mu e_\nu - (-1)^{d_\mu d_\nu} e_\nu e_\mu = C_{\mu\nu}^\rho e_\rho,$$

where  $d_\mu$  is the Grassmann parity of the generator  $e_\mu$  and  $d_\mu = 0$ , if  $\mu = 1, \dots, n$  and  $d_\mu = 1$ , if  $\mu = n + 1, \dots, n + m$ .

For the generators of the algebra  $\mathcal{G}$  the graded modified Yang-Baxter equation (19) at  $\alpha = 1$  takes the form

$$R(\{R e_\xi, e_\gamma\}) + R(\{e_\xi, R e_\gamma\}) - \{R e_\xi, R e_\gamma\} = \{e_\xi, e_\gamma\}$$

which can equivalently be rewritten as follows:

$$(R_\xi^\beta C_{\beta\gamma}^\alpha R_\alpha^\mu + R_\gamma^\beta C_{\xi\beta}^\alpha R_\alpha^\mu - R_\gamma^\beta R_\xi^\alpha C_{\alpha\beta}^\mu) e_\mu = C_{\xi\gamma}^\mu e_\mu. \quad (\text{A.2})$$

Let us define an invariant supersymmetric non-degenerate bi-linear form associated with some representation of  $\mathcal{G}$

$$\langle x, y \rangle := \text{str}(xy) \quad \text{for any } x, y \in \mathcal{G}$$

using which one can introduce the dual basis  $e^\mu$  in superalgebra  $\mathcal{G}$

$$\langle e^\mu, e_\nu \rangle = \delta_\nu^\mu$$

and the supermetric

$$\eta_{\mu\nu} = \langle e_\mu, e_\nu \rangle, \quad \eta^{\mu\nu} = \langle e^\mu, e^\nu \rangle, \quad \eta^{\mu\nu} = -(-1)^{d_\mu} \eta^{\nu\mu} = (\eta_{\nu\mu})^{-1}$$

by which one can raise and lower indices as follows:

$$e_\alpha = \eta_{\alpha\beta} e^\beta, \quad e^\alpha = (-1)^{d_\alpha} \eta^{\alpha\beta} e_\beta = e_\beta \eta^{\beta\alpha}, \quad \eta_{\alpha\nu} C_{\gamma\mu}^\nu \eta^{\beta\mu} = C_{\alpha\gamma}^\beta. \quad (\text{A.3})$$

Note that supermetric  $\eta_{\mu\nu}$  and  $R_\nu^\mu$  are even matrices, i.e., for any their nonzero entry one has  $d_\mu + d_\nu = 0$ .

Now we take the graded tensor product of both sides of eq. (A.2) with  $e^\xi \otimes e^\gamma$  and using relations (A.3) rewrite it in the following form:

$$\begin{aligned} &((-1)^{d_\lambda} R^{\nu\beta} R^{\alpha\mu} C_{\alpha\beta}^\lambda + (-1)^{d_\mu} R^{\lambda\beta} R^{\alpha\mu} C_{\beta\alpha}^\nu \\ &\quad - (-1)^{d_\mu} R^{\lambda\beta} R^{\nu\alpha} C_{\alpha\beta}^\mu) e_\mu \otimes e_\nu \otimes e_\lambda = \eta^{\nu\xi} \eta^{\lambda\gamma} C_{\xi\gamma}^\mu e_\mu \otimes e_\nu \otimes e_\lambda, \end{aligned} \quad (\text{A.4})$$

where

$$R^{\alpha\beta} = (-1)^{d_\alpha} \eta^{\alpha\gamma} R_\gamma^\beta, \quad R_\alpha^\beta = \eta_{\alpha\gamma} R^{\gamma\beta}.$$

Here we use the graded tensor product

$$(e_\mu \otimes e_\nu)(e_\lambda \otimes e_\xi) = (-1)^{d_\nu d_\lambda} (e_\mu e_\lambda \otimes e_\nu e_\xi), \quad (\text{A.5})$$

$$(e_\mu \otimes e_\nu)_{j_1 j_2}^{i_1 i_2} = (e_\mu)_{i_1 j_1} (e_\nu)_{i_2 j_2} (-1)^{d_{i_2}(d_{i_1} + d_{j_1})}, \quad (\text{A.6})$$

where  $d_i(d_j)$  means the Grassmann parity of the row (column) of the supermatrix element  $(e_\mu)_{ij}$  [15] and one has  $d_i + d_j = d_\mu$  for any nonzero  $(e_\mu)_{ij}$ .

Assume further that the operator  $R$  is graded antisymmetric, i.e.,

$$R^{\mu\nu} = -(-1)^{d_\mu} R^{\nu\mu} \iff \langle Rx, y \rangle = -\langle x, Ry \rangle$$

and define two  $(n+m) \times (n+m)$  matrices

$$\tilde{r} = R^{\alpha\beta} e_\alpha \otimes e_\beta, \quad t = (-1)^{d_\alpha} \eta^{\alpha\beta} e_\alpha \otimes e_\beta,$$

where  $\tilde{r}$  is antisymmetric,  $\tilde{r}_{j_1 j_2}^{i_1 i_2} = -\tilde{r}_{j_2 j_1}^{i_2 i_1}$  and  $t$  is the tensor Casimir element invariant with respect to the adjoint action

$$[t, x \otimes 1 + 1 \otimes x] = 0 \quad \text{for any } x \in \mathcal{G}.$$

Note that  $\tilde{r}$  and  $t$  are even matrices, i.e.,  $d_{i_1} + d_{i_2} + d_{j_1} + d_{j_2} = 0$  for any  $\tilde{r}_{j_1 j_2}^{i_1 i_2} \neq 0$  or  $t_{j_1 j_2}^{i_1 i_2} \neq 0$ .

Defining triple graded tensor products

$$\begin{aligned}\tilde{r}_{12} &= R^{\alpha\beta}(e_\alpha \otimes e_\beta \otimes 1), & \tilde{r}_{13} &= R^{\alpha\beta}(e_\alpha \otimes 1 \otimes e_\beta), & \tilde{r}_{23} &= R^{\alpha\beta}(1 \otimes e_\alpha \otimes e_\beta), \\ t_{12} &= \eta^{\beta\alpha}(e_\alpha \otimes e_\beta \otimes 1), & t_{13} &= \eta^{\beta\alpha}(e_\alpha \otimes 1 \otimes e_\beta), & t_{23} &= \eta^{\beta\alpha}(1 \otimes e_\alpha \otimes e_\beta)\end{aligned}$$

and using eqs. (A.2) and (A.5) one can find that eq. (A.4) reads

$$[\tilde{r}_{12}, \tilde{r}_{13} + \tilde{r}_{23}] + [\tilde{r}_{13}, \tilde{r}_{23}] = -[t_{12}, t_{13}] \quad (\text{A.7})$$

that is the tensor form of the graded modified classical Yang-Baxter equation (see e.g. [23]). In order to reproduce the graded classical Yang-Baxter equation with the zero in the r.h.s one needs to introduce a new matrix

$$r = \tilde{r} + t, \quad \tilde{r}_{12} = 1/2(r_{12} - r_{21}), \quad t_{12} = 1/2(r_{12} + r_{21})$$

for which eq. (A.7) takes the form (A.1). Thus, for the case of graded antisymmetric operators  $R$  we have established the equivalence of equations (19) at  $\alpha = 1$  and (A.1) which are, respectively, the operator form of the graded modified classical Yang-Baxter equation and the tensor form of the graded classical Yang-Baxter equation. Note that the former equation is a more general one, since it admits solutions which are not graded antisymmetric.

For completeness we give here the component form of eq. (A.1)

$$\begin{aligned}& r_{k j_2}^{i_1 i_2} r_{j_1 j_3}^k i_3 (-1)^{d_{j_2}(d_{i_3}+d_{j_3})} - r_{k j_3}^{i_1 i_3} r_{j_1 j_2}^k i_2 (-1)^{d_{i_2}(d_{i_3}+d_{j_3})} \\ & + r_{j_1 k}^{i_1 i_3} r_{j_2 j_3}^{i_2 k} (-1)^{d_{i_2}(d_{i_1}+d_{j_1})} - r_{j_2 k}^{i_2 i_3} r_{j_1 j_3}^{i_1 k} (-1)^{d_{j_2}(d_{i_1}+d_{j_1})} \\ & + r_{j_1 k}^{i_1 i_2} r_{j_2 j_3}^k i_3 - r_{k j_3}^{i_2 i_3} r_{j_1 j_2}^{i_1 k} = 0,\end{aligned}$$

where eq. (A.6) is used when calculating the triple tensor products.

## Appendix B

In this Appendix we investigate bi-linear bracket (21) and establish conditions on the  $R$ -matrix and its graded antisymmetric part which are necessary in order the bracket (21) be the Poisson bracket. Therefore, we need to verify the Jacobi identities for any  $f, g$  and  $h$  in  $\mathfrak{g}$

$$\begin{aligned}(-1)^{d_h d_g} \{h, \{f, g\}_2\}_2 + \text{c. p.} &= -1/4 (-1)^{d_h d_g} < [\mathbb{O}, \nabla(\{f, g\}_2)] \cdot \\ & R \left( (\nabla h)^{*(d_g+d_f)} \mathbb{O}^{*(d_f+d_g+d_h)} + \mathbb{O}^{*(d_g+d_f)} (\nabla h)^{*(d_g+d_f)} \right) \\ & - R \left( \nabla(\{f, g\}_2) \mathbb{O}^{*(d_g+d_f)} + \mathbb{O} \nabla(\{f, g\}_2) \right) \cdot \\ & [\mathbb{O}^{*(d_g+d_f)}, (\nabla h)^{*(d_g+d_f)}] > + \text{c. p.} = 0,\end{aligned} \quad (\text{B.1})$$

where

$$\nabla(\{f, g\}_2) = -1/4 \left[ \nabla g R((\nabla f)^{*(d_g)} \mathbb{O}^{*(d_f+d_g)} + \mathbb{O}^{*(d_g)} (\nabla f)^{*(d_g)}) \right]$$

$$\begin{aligned}
& +R(\mathbb{O}\nabla g + \nabla g\mathbb{O}^{*(d_g)}) (\nabla f)^{*(d_g)} \\
& +\nabla g R^\dagger((\nabla f)^{*(d_g)}\mathbb{O}^{*(d_f+d_g)} - \mathbb{O}^{*(d_g)}(\nabla f)^{*(d_g)}) \\
& +R^\dagger(\mathbb{O}\nabla g - \nabla g\mathbb{O}^{*(d_g)}) (\nabla f)^{*(d_g)} \\
& -(-1)^{d_f d_g} \left( R(\nabla f\mathbb{O}^{*(d_f)} + \mathbb{O}\nabla f) (\nabla g)^{*(d_f)} \right. \\
& +\nabla f R((\nabla g)^{*(d_f)}\mathbb{O}^{*(d_f+d_g)} + \mathbb{O}^{*(d_f)}(\nabla g)^{*(d_f)}) \\
& +\nabla f R^\dagger((\nabla g)^{*(d_f)}\mathbb{O}^{*(d_f+d_g)} - \mathbb{O}^{*(d_f)}(\nabla g)^{*(d_f)}) \\
& \left. +R^\dagger(\nabla f\mathbb{O}^{*(d_f)} - \mathbb{O}\nabla f) (\nabla g)^{*(d_f)} \right) \Big]. \tag{B.2}
\end{aligned}$$

Inserting (B.2) into (B.1) after tedious but straightforward calculations we get for the Jacobi identities

$$(-1)^{d_f(d_h+d_g)} < [\mathbb{O}, h] \left( [R^\dagger F_-, R^\dagger G_-] + [RF_+, RG_+] - R([F_+, G_+]_R) \right) > +\text{c.p.} = 0,$$

where we introduce the notation

$$\begin{aligned}
F_\pm &= \mathbb{O}^{*(d_h)}(\nabla f)^{*(d_h)} \pm (\nabla f)^{*(d_h)}\mathbb{O}^{*(d_h+d_f)} \\
G_\pm &= \mathbb{O}^{*(d_h+d_f)}(\nabla g)^{*(d_h+d_f)} \pm (\nabla g)^{*(d_h+d_f)}\mathbb{O}^{*(d_h+d_f+d_g)}
\end{aligned}$$

for brevity. For any linear map  $R$  and its graded antisymmetric part  $A = 1/2(R - R^\dagger)$  one has the identity

$$\begin{aligned}
(-1)^{d_f(d_h+d_g)} &< [\mathbb{O}, h] [R^\dagger F_-, R^\dagger G_-] > +\text{c.p.} \\
&= (-1)^{d_f(d_h+d_g)} < [\mathbb{O}, h] \left( 4/3([AF_-, AG_-] - A([F_+, G_+]_A) \right. \\
&\quad \left. - ([RF_-, RG_-] - R([F_-, G_-]_R)) \right) > +\text{c.p.} \tag{B.3}
\end{aligned}$$

Now using (B.3) and the following identity

$$(-1)^{d_f(d_h+d_g)} < [\mathbb{O}, h] \left( [F_-, G_-] + 3[F_+, G_+] \right) > +\text{c.p.} = 0,$$

which can directly be verified, we finally rewrite the Jacobi identities with an arbitrary parameter  $\alpha$  as follows:

$$\begin{aligned}
(-1)^{d_f(d_h+d_g)} &< [\mathbb{O}, h] \left( [RF_+, RG_+] - R([F_+, G_+]_R) + \alpha[F_+, G_+] \right. \\
&\quad - ([RF_-, RG_-] - R([F_-, G_-]_R) + \alpha[F_-, G_-] \\
&\quad \left. + 4/3([AF_-, AG_-] - A([F_-, G_-]_A) + \alpha[F_-, G_-]) \right) > +\text{c.p.} = 0.
\end{aligned}$$

Now it is obvious that the Jacobi identities are satisfied if  $R$  and its graded antisymmetric part  $A$  obey the graded modified Yang-Baxter equation (19) with the same  $\alpha$ .



## References

- [1] A.V. Mikhailov, Pisma Zh. Eksp. Teor. Fiz. **30** (1979) 443.
- [2] K. Ueno and K. Takasaki, "Toda Lattice Hierarchy", *Adv. Stud. in Pure Math.* **4** (1984) 1
- [3] M.A. Olshanetsky, "Supersymmetric two-dimensional Toda lattice", *Commun. Math. Phys.* **88** (1983) 63.
- [4] D.A. Leites, M.V. Saveliev, and V.V. Serganova, "Embeddings of  $osp(1|2)$  and the associated nonlinear supersymmetric equations", in Group Theoretical Methods in Physics, Vol. **I** (Yurmala, 1985), VNU Sci. Press, Utrecht, 1986, 255.
- [5] V.A. Andreev, "Odd bases of Lie superalgebras and integrable systems", *Theor. Math. Phys.* **72** (1987) 758.
- [6] K. Ikeda, "A supersymmetric extension of the Toda lattice hierarchy", *Lett. Math. Phys.* **14** (1987) 321.
- [7] J. Evans and T. Hollowood, "Supersymmetric Toda field theories", *Nucl. Phys.* **B 352** (1991) 723.
- [8] O. Lechtenfeld and A.S. Sorin, "Fermionic flows and tau function of the  $N = (1|1)$  superconformal Toda lattice hierarchy", *Nucl. Phys.* **B 557** (1999) 535; "Hidden  $N = (2|2)$  supersymmetry of the  $N = (1|1)$  supersymmetric Toda lattice hierarchy", *J. Nonlin. Math. Phys.* **8** (2001) 183; "A note on fermionic flows of the  $N = (1|1)$  supersymmetric Toda lattice hierarchy", *J. Nonlin. Math. Phys.* **11** (2004) 294.
- [9] V.G. Kadyshevsky and A.S. Sorin, "Supersymmetric Toda lattice hierarchies", In "Integrable Hierarchies and Modern Physical Theories" (Eds. H. Aratyn and A.S. Sorin), Kluwer Acad. Publ., Dordrecht/Boston/London, (2001) 289, nlin.SI/0011009.
- [10] V.V.Gribanov, V.G. Kadyshevsky, and A.S. Sorin, "Periodic supersymmetric Toda lattice hierarchy", *Czech. J. Phys.* **54** (2004) 1289.
- [11] V.V.Gribanov, V.G. Kadyshevsky, and A.S. Sorin, "Generalized Fermionic Discrete Toda Hierarchy", *Discrete Dynamics in Nature and Society* **2004:1** (2004) 113.
- [12] V.B. Derjagin, A.N. Leznov, and A.S. Sorin, "The solution of the  $N = (0|2)$  superconformal f-Toda lattice", *Nucl.Phys.* **B 527** (1998) 643.
- [13] G. Carlet, "The Hamiltonian Structures of the Two-Dimensional Toda Lattice and R-matrices", math-ph/0403049.
- [14] V.G. Kadyshevsky and A.S. Sorin, " $N = (1|1)$  supersymmetric dispersionless Toda lattice hierarchy", *Theor. Math. Phys.* **132** (2002) 1080; "Continuum limit of the  $N = (1|1)$  supersymmetric Toda lattice hierarchy", JHEP Proceedings, PrHEP unesp2002, Workshop on Integrable Theories, Solitons and Duality, 1-6 July 2002, Sao Paulo, Brazil.

- [15] P.P. Kulish and E.K. Sklyanin, "On the solution of the Yang-Baxter equation", *J. Sov. Math.* **19** (1980) 1596; *Zap. Nauch. Sem. LOMI* **95** (1980) 129 (in Russian).
- [16] V.V.Gribanov, V.G. Kadyshesky, and A.S. Sorin, "Hamiltonian structures of fermionic two-dimensional Toda lattice hierarchies", nlin.SI/0505039.
- [17] H. Aratyn and K. Bering, "Compatible Poisson Structures of Toda Type Discrete Hierarchy", nlin.SI/0402014.
- [18] M.A. Semenov-Tyan-Shanskii, "What a Classical R-matrix Is", *Functional. Anal. i Prilozhen.* **17** (1983) 17 (in Russian); *Functional Anal. Appl.* **17** (1983) 259.
- [19] C.M. Yung, "The modified classical Yang-Baxter equation and supersymmetric Gel'fand-Dikii brackets", *Mod. Phys. Lett.* **8** (1993) 129.
- [20] W. Oevel and O. Ragnisco, "R-matrices and Higher Poisson Brackets for Integrable systems", *Phys. A* **161** (1989) 181.
- [21] L.-C. Li and S. Parmentier, "Nonlinear Poisson Structures and r-matrices", *Comm. Math. Phys.* **125** (1989) 545.
- [22] K. Kajiwara and J. Satsuma, "The conserved quantities and symmetries of the two-dimensional Toda lattice hierarchy", *J. Math. Phys.* **32** (1991) 506.
- [23] A.P. Isaev, "Quantum groups and Yang-Baxter equations", *Sov. J. Part. Nucl.* **26** (1995) 501; *Fiz. Elem. Chastits i At. Yadra* **26** (1995) 1204 (in Russian).